Structure of exotic three-body systems

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Abstract. The classification of large halos formed by two identical particles and a core is systematically addressed according to interparticle distances. The root-mean-square distances between the constituents are described by universal scaling functions obtained from a renormalized zero-range model. Applications for halo nuclei, ^{11}Li and ^{14}Be, and for atomic ^{4}He$_3$ are briefly discussed. The generalization to four-body systems is proposed.

1 Introduction

The properties of large three-body systems present universal behavior as the Thomas-Efimov limit is approached [1]. In this situation the sizes of the two and three body systems are much larger than the interaction range and the wave function almost everywhere is an eigenstate of the free Hamiltonian. Much of the physics of these systems can be studied using a zero-range interaction effective in s-wave. No scale is involved in this drastic situation and the momentum space Faddeev equations are scale invariant for vanishing two and three-body energies. The scale invariance is broken by finite values of two-body scattering lengths and three-body binding energy, which are the scales of the corresponding three-body systems [2]. Dimensionless ratios of three-body observables are then expressed by scaling functions which depends only on ratios of two (bound or virtual) and three-body energies [3]. The scaling functions can be obtained from the solution of regularized Faddeev equations.

The sizes of halo three-body systems (where two particles are identical) are functions of few physical scales. A classification scheme for these systems [4, 5, 6] ordered by their sizes is reviewed. For a given three-body binding energy the most compact system is the Borromean (single pairs are unbound) while the all-bound (all single pairs form bound states) is the largest one. The neutron-neutron mean square radius in examples of Borromean halo nuclei which are

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known experimentally [7] are briefly discussed. The extension of these ideas to four-body systems is proposed.

2 Scale invariance in three-boson systems

The collapse of a three-boson system when the two-body interaction range goes to zero (Thomas effect, see [1]) demands one new physical scale to stabilizes the three-boson binding energy (E_3). The regulated trimer bound state integral equation in units of $\hbar = m = 1$ (m is the boson mass) is written as

$$f(q) = 2\pi\tau(E_3 - \frac{3}{4}q^2)\int_0^\infty k^2 dk \ f(k)\int_{-1}^1 dz \left[G_0(E_3) - G_0(-\mu_{(3)}^2)\right], \qquad (1)$$

where $G_0^{-1}(E) = E - q^2 - k^2 - qkz$, and the renormalized two-boson scattering amplitude is $\tau^{-1}(x) = 2\pi^2\left[a^{-1} - \sqrt{-x}\right]$. Without regularization, Eq. (1) is the Skorniakov and Ter-Martirosian (SKTM) equation derived long ago [8] which is scale invariant when the scattering length $|a|$ tends to infinity and $E_3 = 0$. The second term in Eq. (1) brings the physical scale, $\mu_{(3)}$, to the three-boson system [2] avoiding the Thomas collapse. The equivalence of Thomas and Efimov effects is seen in units of $\mu_{(3)} = 1$, which means $|a|\mu_{(3)} \to \infty$ either for $|a| \to \infty$ (Efimov limit) or for $\mu_{(3)} \to \infty$ (Thomas limit).

The sensitivity of three-boson S-wave observables to the short-range part of the interaction in weakly bound systems is parameterized through the value of the trimer binding energy which corresponds to the scale $\mu_{(3)}$. Three-boson S-wave observables are strongly correlated to the trimer energy in a general universal form [3, 9]:

$$\mathcal{O}_3(E, E_3, a) = |E_3|^\eta \mathcal{F}_3\left(E/E_3, a\sqrt{|E_3|}\right) , \qquad (2)$$

where $\mathcal{O}_3$ can represent a scattering amplitude at an energy E or an excited trimer energy (the dependence on E does not appear in this case). The exponent η gives the correct dimension to $\mathcal{O}_3$. Eq. (1) is renormalization group (RG) invariant with its kernel being a solution of a Callan-Symanzik differential equation as function of a sliding $\mu_{(3)}$. In that way E_3 and three-body observables are independent of the subtraction point (see ref. [9] for a discussion of the RG invariance in three-body systems).

3 Classification of three-body halos and universal scalings

The sizes of halo three-body systems (with two identical particles (α) and a distinct one (β)) are functions of few physical scales. A classification scheme for these systems as Borromean (single pairs are unbound), Tango (only the $\alpha\alpha$ pair form a bound state), Samba (only the $\alpha\beta$ pair is bound) and all-bound (all single pairs form bound states), ordered by their sizes is discussed below.

The typical lengths of the three-body halo systems are given by scaling functions for the mean-square separation distances written according to Eq.(2). The

scaling functions for these radii are:

$$\sqrt{\langle r_{\alpha\gamma}^2 \rangle |E_3|} = \mathcal{R}_{n\gamma}\left(\pm\sqrt{E_{\alpha\alpha}/E_3}, \pm\sqrt{E_{\alpha\beta}/E_3}, A\right), \tag{3}$$

$$\sqrt{\langle r_{\gamma}^2 \rangle |E_3|} = \mathcal{R}_{\gamma}^{cm}\left(\pm\sqrt{E_{\alpha\alpha}/E_3}, \pm\sqrt{E_{\alpha\beta}/E_3}, A\right), \tag{4}$$

where $\langle r_{\alpha\gamma}^2 \rangle$ and $\langle r_{\gamma}^2 \rangle$ are, respectively, the mean square relative and center of mass distances. The mass ratio is $A = m_\beta/m_\alpha$ and γ is α or β. The + or - signs represent bound or virtual two-body subsystems, respectively. $E_{\alpha\alpha}$ and $E_{\alpha\beta}$ are the $\alpha\alpha$ and $\alpha\beta$ two-body energies.

For a given energy E_3 and identical particles, the effective interaction in Eq. (1) has a weaker strength for $a < 0$ (virtual two-body system) than for $a > 0$ (bound two-body system). Therefore, for a Borromean trimer Eq. (1) should have a larger value of μ_3 than the corresponding one for an all-bound system, in order to keep the binding fixed with a weaker interaction. The spectator function, $f(q)$, extends to large momentum for a Borromean system. The trimer wave function for zero total angular momentum is

$$\Psi(q,p) = \frac{f(|\boldsymbol{q}|) + f(|\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}|) + f(|\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}|)}{E_3 - \frac{3}{4}q^2 - p^2}, \tag{5}$$

which implies in a more compact spatial configuration for a Borromean trimer in comparison to the all-bound one. The Jacobi relative momenta are $\boldsymbol{p}$ for the pair and $\boldsymbol{q}$ for the spectator particle. In terms of the scaling functions the radii come as,

$$\mathcal{R}_{\alpha\alpha}\left(-\sqrt{E_{\alpha\alpha}/E_3}\right) < \mathcal{R}_{\alpha\alpha}\left(\sqrt{E_{\alpha\alpha}/E_3}\right), \tag{6}$$

and the separation distances obeys $\sqrt{\langle r_{\alpha\alpha}^2 \rangle |E_3|}\big|_B < \sqrt{\langle r_{\alpha\alpha}^2 \rangle |E_3|}\big|_A$, where the labels B and A correspond to Borromean and All-bound systems, respectively. For the same reasons that led to Eq. (6), it is also valid that $\sqrt{\langle r_{\alpha}^2 \rangle |E_3|}\big|_B < \sqrt{\langle r_{\alpha}^2 \rangle |E_3|}\big|_A$. The zero-range model applied to atomic ^{4}He$_3$ provides a qualitative understanding of the radii results of realistic calculations for the ground and excited states [10] with an estimation of $\sqrt{\langle r_{\alpha\alpha}^2 \rangle} = \mathcal{C}\sqrt{\hbar^2/[m_\alpha(|E_3 - E_{\alpha\alpha}|)]}$ with $0.6 < \mathcal{C} < 1$ [6]. Also the excited trimer state energy of ^{4}He$_3$, E_3^*, has a scaling behavior written as

$$\sqrt{|E_3^* - E_{\alpha\alpha}|} = \sqrt{|E_3|}\,\mathcal{E}\left(\pm\sqrt{E_{\alpha\alpha}/E_3}\right), \tag{7}$$

which is consistent with results from realistic calculations (see [3, 9] and references therein). The threshold for the appearance of an excited Efimov trimer state from the second energy sheet is $|E_3| = 6.9/a^2$ [9, 11] in units of ($\hbar = m_\alpha = 1$). The scaling behavior was extended to the complex energy plane and for three-boson Borromean systems, the excited Efimov state turns into a resonance when the virtual two-boson virtual state energy is decreased [12].

The generalization of the reasonings leading to Eq. (6) to the $\alpha\alpha\beta$ system gives the qualitative classification of the different three-body systems in respect

to sizes. The effective interaction is weaker when a pair has a virtual state than when the pair is bound, and the three-body system has to shrink to keep the binding energy unchanged if a pair which is bound turns to be virtual. Therefore it is reasonable to expect that

$$\mathcal{R}_{\alpha\gamma}\left(-\sqrt{E_{\alpha\alpha}/E_3}, -\sqrt{E_{\alpha\beta}/E_3}, A\right) < \mathcal{R}_{\alpha\gamma}\left(\sqrt{E_{\alpha\alpha}/E_3}, -\sqrt{E_{\alpha\beta}/E_3}, A\right)$$
$$< \mathcal{R}_{\alpha\gamma}\left(-\sqrt{E_{\alpha\alpha}/E_3}, \sqrt{E_{\alpha\beta}/E_3}, A\right) < \mathcal{R}_{\alpha\gamma}\left(\sqrt{E_{\alpha\alpha}/E_3}, \sqrt{E_{\alpha\beta}/E_3}, A\right), \quad (8)$$

which was checked numerically [6]. An analogous relation is valid for $\mathcal{R}_\gamma^{cm}$. The dimensionless products (we are using units of $\hbar = m_\alpha = 1$) $\sqrt{\langle r_{\alpha\beta}^2\rangle|E_3|}$ and $\sqrt{\langle r_{\alpha\alpha}^2\rangle|E_3|}$ increase from *Borromean, Tango, Samba* and to *All-Bound* configurations, systematizing the classification scheme proposed in Ref. [5] for weakly bound three-body systems.

   A three-body model applied to light exotic nuclei [6] compares qualitatively well with the existent experimental data for the neutron-neutron separation distance in the neutron-halo of ^{11}Li and ^{14}Be [7]. Therefore, the neutrons of the halo have a large probability to be found outside the interaction range and the low-energy properties of the halo are, to a large extend, universal as long as few physical input scales are fixed in the model. An insight into the structure of halo nuclei can be found even considering the limitations of the model. The finite size of the core and consequently the antisymmetrization of the total nuclear wave function, are both missing in this simplified description. However, in examples where the neutrons in the halo tend to be more and more weakly bound with virtual or bound subsystems near the scattering threshold, the scaling relations apply for the halo properties, as the above limitations are less important.

4 Classification of four-body halos and universal scalings

The four-boson system has two Faddeev-Yakubovsky (FY) independent amplitudes and within a zero-range model, they are reduced to spectator amplitudes depending on two Jacobi momenta. The two spectator FY reduced amplitudes satisfy a coupled set of integral equations generalizing the SKTM equation for three bosons. The set of coupled integral equations needs regularization, and one recognizes that the resolvent of the immersed three-boson subsystem carries the scale $\mu_{(3)}$. Other terms are present and require regularization. We introduce a scale $\mu_{(4)}$ such that the four-body free Green's function $G_0(E_4)$ are substituted by $G_0(E_4) - G_0(-\mu_{(4)}^2)$ [13] in a direct generalization of Eq. (1) as suggested by [2].

   The momentum scales in the FY equations for the reduced amplitudes are only a^{-1}, $\mu_{(3)}$ and $\mu_{(4)}$. In this case, the tetramer binding energy depends on the momentum scales as

$$E_4 = \mu_{(3)}^2\, \overline{\mathcal{E}}_4\left(\mu_{(4)}/\mu_{(3)}, a\mu_{(3)}\right). \quad (9)$$

For $a = \infty$ the trimer binding energy from the solution of Eq. (1) is $E_3 = -0.0093\mu_{(3)}^2$ [9], which simplifies Eq. (9) remaining only the dependence on the

ratio $\mu_{(4)}/\mu_{(3)}$ in E_4:

$$E_4 = E_3 \; \mathcal{E}_4 \left(\mu_{(4)}/\mu_{(3)} \right) . \tag{10}$$

If $\mathcal{E}_4$ is independent on the regularization scale $\mu_{(4)}$ for $\mu_{(4)}/\mu_{(3)} >> 1$ the four-body scale is not important. We solved numerically the FY equations up to large values of $\mu_{(4)}/\mu_{(3)} \sim 20$ [13].

The tetramer ground state binding energy was calculated for different values of the ratio $\mu_{(4)}/\mu_{(3)}$ with $a = \infty$. The ratio E_4/E_3 depends strongly $\mu_{(4)}/\mu_{(3)}$. For equal three and four-body scales, i.e., $\mu_{(4)}/\mu_{(3)} = 1$, the tetramer binding energy is $E_4 = 5 \; E_3$, agreeing with the angular coefficient of the Tjon line [14], $E_4 = 4.72(E_3 + 2.48)$ MeV (E_4 is the ^{4}He energy and E_3 the triton one). Also, a recent calculation [15] of the four-boson system with a two-body zero-range force and a repulsive three-body potential to stabilize the trimer against collapse, E_4 scales as $\sim 5 \; E_3$. Our result for $\mu_{(4)} = \mu_{(3)}$ agrees with both. However, for $\mu_{(4)}/\mu_{(3)} = 20$ we found that the ratio between the tetramer and trimer energies is about 78, indicating the independent effect of the four-body scale.

The present results suggest that the general scaling of S-wave three-boson observables with the physical scales, Eq. (2), may be generalized to four-boson S-wave observables. The effect of the short-range dynamics in an observable comes through the values of the scattering length, trimer and tetramer binding energies, associated with $\mu_{(3)}$ and $\mu_{(4)}$, respectively. In this case, a S-wave four-boson observable will be strongly correlated to a, E_3 and E_4:

$$\mathcal{O}_4(E, E_4, E_3, a) = |E_4|^\eta \mathcal{F}_4 \left(E/E_4, E_3/E_4, a\sqrt{|E_4|} \right) , \tag{11}$$

where $\mathcal{O}_4$ represents either a scattering amplitude at energy E, or an excited tetramer energy or some observable related to the tetramer (the dependence on E does not appear in these cases). The exponent η gives the correct dimension to $\mathcal{O}_4$. For sizes one could think that the relation

$$\mathcal{R}_{\alpha\alpha} \left(\sqrt{E_3/E_4}, -\sqrt{E_{\alpha\alpha}/E_4} \right) < \mathcal{R}_{\alpha\alpha} \left(\sqrt{E_3/E_4}, \sqrt{E_{\alpha\alpha}/E_4} \right), \tag{12}$$

and the analogous for the distances of the particles to the center of mass would be valid. This indicates that is possible to envisage a generalized classification scheme based on sizes including weakly-bound four-body systems.

5 Outlook and Conclusions

The classification scheme of large halos formed by two identical particles and a core is reviewed and addressed systematically according to their sizes. The root-mean-square distances between the constituents are described by universal scaling functions. For a given three-body system and total energy, the Borromean configuration is the most compact. Applications to halo nuclei, ^{11}Li and ^{14}Be, and for atomic ^{4}He$_3$ were briefly discussed.

The generalization of these concepts to four-body systems is proposed. We have shown that for a zero-range two-body interaction with an infinite scattering length and a fixed trimer ground state binding energy, a four boson momentum scale is evidenced in the calculation of tetramer binding energies in

three-dimensions. The intensity of the effective interaction that composes the kernel of the reduced FY spectator equations depends on the dimer energy and trimer, therefore it is reasonable that a tetramer becomes more compact for a given four-body energy if the two and three-body binding are decreased. This effect indicates that it may be possible to generalize the classification scheme of weakly-bound three-body systems to four-body systems.

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